Service Woes at *Speedy Delivery*: Finding the Shortest Route

Many important optimization problems can best be analyzed by means of a graphical or network representation. A network consists of a set of points, called nodes (vertices), which are connected by segments, called arcs (edges). Figure 1 shows a simple exa mple:

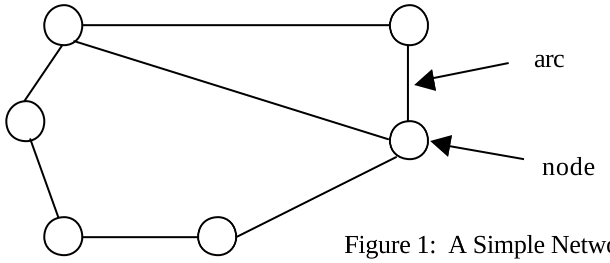


Figure 1: A Simple Network

In practical applications, the nodes often represent geographic points like cities, intersections, railroad stops, pipeline connections, or individual locations. The arcs often represent links between nodes, for example, roads between cities. The arcs can be undirected (two way) or directed (one way). Sometimes the arcs are weighted with a numerical value representing distance, travel time, or cost when traversing the arc. A basic problem involving networks is to find the shortest path between two given nodes.

<u>Sample Problem</u>

The Speedy Delivery Company has the delivery area represented by the network shown in Figure 2:

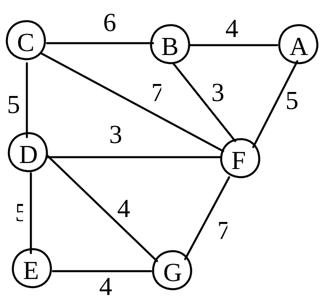
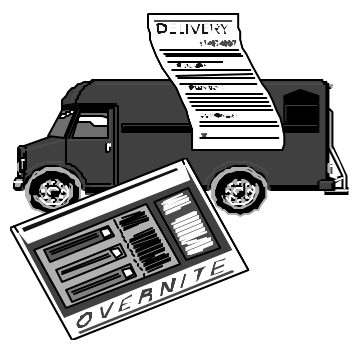


Figure 2: Speedy Delivery's Service Area



The numerical weightings are the average driving times in minutes between two pick-up locations. For example, on average it should take a driver 4 minutes to drive from location A to location B.

1 On average how long should it take the driver to travel from A to B to C (path ABC)?______

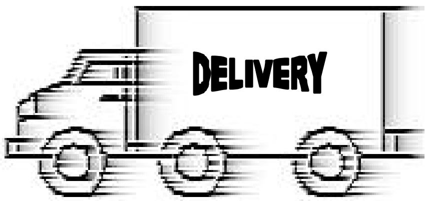
Node A represents the company headquarters and node E is the location of the company's largest customer. Mr. Harry Upp, the dispatcher for Speedy Delivery, wants to find the quickest route from headquarters (A) to the largest customer (E).

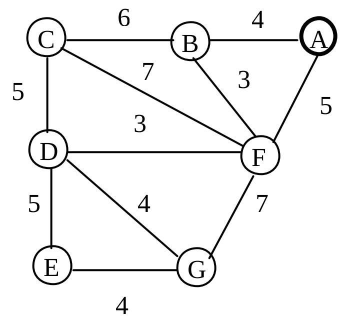
2 What route from A to E do you think requires the least time? ____________________________________
Ask other students nearby for their answer for the trip from A to E? What path did they take?
____.

As you can see, there are a variety of paths and times from A to E. There is a method to find the shortest path, which is the quickest time in this problem. Such methods are called algorithms. Dijkstra's Algorithm is a shortest path algorithm. Its steps are:

1. Circle the starting node (vertex). Examine all arcs (edges) that have that node as an endpoint. Darken the arc with the shortest length and circle the node at the other endpoint of the darkened arc.
2. Examine all uncircled nodes that are adjacent to the circled nodes in the graph.
3. Using only circled nodes and darkened arcs between the nodes that are circled, find lengths of *each path* from starting point to those nodes in Step 2. Choose the node and arc that yield the shortest path. Circle this node and darken this arc. Ties are broken arbitrarily (if two or more paths have the same total length then you can choose either of them).
4. Repeat steps 2 and 3 until all nodes are circled. The darkened arcs of the graph form the shortest routes from your starting point to every other node in the graph.

In this example (Figure 2), your starting point is A. Therefore, it is necessary for you to circle it.





3 There are two nodes adjacent to A. What are they?____________

4 What are the two paths from A?_______________

Darken the arc with the shortest time. Circle the node that is at the endpoint of this arc.

5 From this node, what are the two new uncircled adjacent nodes ?_____________

6 List all of the paths starting at A and ending at one of the uncircled adjacent nodes. Find the time for each of these paths. ____
____

Circle the node and darken the arc that would create the shortest path to an uncircled adjacent node.

7 What node did you circle?_________ Which arc did you darken?________

Path AF and path ABF both led to the uncircled node F. AF was chosen over ABF because it was a shorter time to node F. A path through BF to get to F would never be chosen because it is too long. Therefore, strike out BF as a path on your graph.

8 What are the three uncircled nodes adjacent to the node you listed in number **7?____________**

9 What are all the paths starting at A and leading to the uncircled adjacent nodes? List them and the total times for each of these paths. ____
____

Circle the node and darken the arc that would create the shortest path.

10 What node did you circle?_______ Which arc did you darken?__________

Compare your information with the table below.

Circled Node	**Adjacent Nodes (uncircled)**	**Path (from A)**	**Total Time**
1st A	B	AB	4
	F	AF	5
2nd B	F	~~ABF~~	~~7~~
	C	ABC	10
3rd F	D	AFD	8
	G	AFG	12
	C	AFC	12

This is a summary of the steps to be followed to produce the table:

1. List the beginning node on the table and circle it on the graph.
2. List the uncircled adjacent nodes, the paths, and the total time.
3. Identify the shortest total time (4). In the table circle the adjacent node that created the shortest time and circle the path. On the drawing , darken the circle of the node and the path.
4. The node that was just identified has become a circled node in the table. Enter it in the circled node column of the table.
5. Repeat step 2 with the new circled node. Find the shortest total time for ***all*** of the paths in the table that are **not already circled** (used) or **not crossed off** (too long).
6. The next shortest total time is 5. Circle F and underline AF in the table. On the drawing, circle the node and darken in the path. Cross off all other paths which contain F as the uncircled adjacent node (ABF). The path ABF is 7 which is longer than the path which was identified. Strike path BF off the drawing for the same reason.
7. Repeat steps 4 and 5, this time with F as the circled node.
8. The shortest time at this point is 8 and the path is AFD. Circle D in the table and circle the path AFD. On the drawing, darken the circle of the node and darken the path. Cross off any other paths which end at D. (None exist.)

Even though there is no solution to Mr. Upp's problem yet, the following questions can be answered:

11 What is the quickest route from A to B? ______
12 What is the quickest route from A to F? ______
13 What is the quickest route from A to D? ______

The algorithm will not only find the shortest path desired but will also find the shortest path to other nodes in the graph from the designated starting point. Since the solution has not been found, the process must be continued. Steps 4 and 5 must be repeated using the new node.

Circled Node	**Adjacent Nodes (uncircled)**	**Path (from A)**	**Total Time**
1st A	B	AB	**4**
	F	AF	**5**
2nd B	F	~~ABF~~	~~7~~
	C	ABC	10
3rd F	D	AFD	**8**
	G	AFG	12
	C	AFC	12
4th D	___	______	___
	___	______	___
	___	______	___

14 What is the next shortest total time? _______
15 What adjacent node led to that time? _______ This node should be circled in the table as well as on the drawing. Circle the path in the table and the darken the arc on the drawing.
16 Are there any remaining arcs listed that end at the node that was just circled? ______ If so, cross off those arcs on the table as well as in the drawing, because they are no longer viable.
17 Underline the next shortest path in the table, circle the node, and darken the arc. Which node did you circle? _____ Which arc did you darken? _____

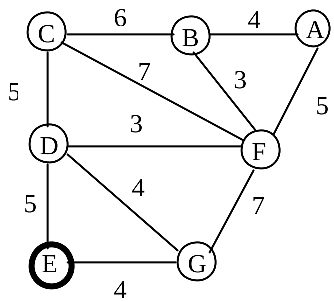
Continuing with the table:

Circled Node	**Adjacent Nodes (uncircled)**	**Path (from A)**	**Total Time**
5th **C**	NONE		
6th ___	_________	_____	_____
7th ___	NONE		

According to the **Shortest Path Algorithm**, there are no uncircled nodes adjacent to C. This means that the shortest path from A to E does not pass through C. Continue with step 3 of the Algorithm and complete the table.
What is unique about the next shortest time? Remember that ties may be broken arbitrarily.
It should also be noted that once the table includes a path to E you are not finished unless you have circled E. ***Once E is circled***, you have determined the shortest path to it. It is not possible to find later some other path from A to E that is shorter.

18 What is the quickest route from A to C? _______________
19 What is the quickest route from A to G? _______________
20 What is the quickest route from A to E? _______________
21 Harry Upp, the dispatcher, needs to have the driver return to headquarters. Should the driver take the same route or a different route in order to arrive at headquarters as quickly as possible? _________________________

22 The driver, I.M. Lait, radios to Mr. Upp that he needs to pick up a package at destination B before he returns to headquarters. Go through the algorithm using E as the starting point and B as the destination. Show the work on the graph below and fill in the table, as in the previous example.



Circled Node	Adjacent Nodes (uncircled)	Path (from E)	Total Time
E			

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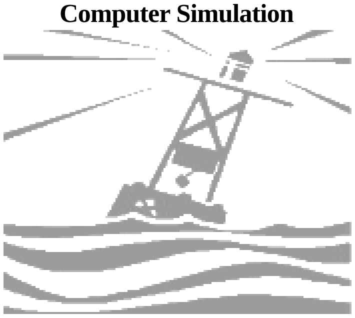

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