Teacher Resources

Integer Programming

Linear programming (LP) involves finding the optimal values for a set of decision variables that are constrained by a system of linear inequalities. The objective can be to minimize cost or maximize profit. Linear programming has a broad range of applications that includes planning production rates for a mix of products, blending gasoline and reducing trim loss when cutting large sheets of paper or steel. In these contexts the decision variables are continuous and the optimal solution must lie at a corner point of the feasible region formed by the system of linear inequalities. There are situations, however, in which the decision variables must assume only integer values. A regional manager deciding how many cars to have at each of her car rental locations knows that the solution must be integer. Planning the number of police officers to have on duty each hour of the day requires that the solution take on integer values. Problems of these types in which the decision domain is restricted to integer values are called integer programming (IP) problems. Problem contexts that involve both integer and continuous decision variables are termed *mixed-integer programming*. Another equivalent term often used is combinatorial optimization. Optimal scheduling and sequencing decisions are often formulated as IP problems.

General purpose IP problems are orders of magnitude more complex to solve than LP problems because the optimal solution need no longer be at a corner point. However, there are IP problems that have structures that facilitate the development of special purpose solution algorithms that are even faster to solve than comparably sized LP problems. Interestingly, there are classes of LP problems in which the matrix structure of the system of linear inequalities guarantees that the optimal solution to the LP problem will have all integer values. The Transportation Problem is in that class. The Transportation Problem involves minimizing the total cost of transporting supplies of a single product type from various locations (e.g., plants or warehouses) to a set of demand points (e.g., retail facilities). The decision variables are how much to ship from point A to point B and the constraints are the location-specific available supplies and demand.

An especially powerful IP modeling concept is the use of integer decision variables that take on only the values of 0 or 1. These 0-1 decision variables, also called indicator variables, are used to describe yes-no decisions. Should a facility be built at this location or not? Should the corporation invest in this project or not? The Assignment Problem is a specific example of using only 0-1 variables. The Assignment Problem involves assigning, for example, people (or machines) to a set of mutually exclusive set of jobs (or tasks) so as to minimize the total cost (or time) of completing these jobs. The decision variable Y_{a_1} takes on the value "1" if person *a* is assigned to do job *1* and equals 0 if he is not assigned to that job.

The Integer Programming student activity, case studies and homework summarized in this lesson activity are limited to 0-1 decision variables. An earlier activity, Manpower Planning at Pizza ? includes an extension that explores the issue of

integer decision variables that are not restricted to 0-1 values. The examples in this activity include a) investment portfolio selection, b) dividing (assigning) a set of season tickets amongst a group of buyers, c) facility location problems and d) steel production planning.

K. L. Hoffman and M. Padberg, "Combinatorial and Integer Optimization," in S. I. Gass and C. M. Harris, editors, Encyclopedia of Operations Research and Management Science, Centennial Edition, (Boston: Kluwer Academic Publishers, 2001), 94-102.

Case Studies

Grantham, Mayo, Van Otterloo and Company. Grantham, Mayo, Van Otterloo and Company LLC (GMO) is an investment-management firm that offers a wide range of mutual funds to clients world-wide. Most of its clients are pension funds, educational endowments, foundations, and similar organizations. GMO makes extensive use of statistical methods, simulation, and optimization techniques to manage billions of dollars of assets.

In 1996, GMO's quantitative group had accumulated $15 billion in assets, but the large number of different stocks in the portfolio, as well as the huge number of transactions required to rebalance these assets, had many clients uneasy about GMO's ability to manage the portfolio. Moreover, the large numbers of different stocks and transactions were straining GMO's operations and increasing the costs of maintaining these assets.

In response, GMO developed and put into practice a mixed-integer-programming model that includes the use of 0-1 decision variables to construct portfolios that are close to target portfolios in terms of liquidity, risk, turnover, and expected return, yet include fewer distinct stocks and require far fewer stock transactions. Typically, these portfolios consist of several subportfolios that represent different investment philosophies. An important advantage of such a structure is that in optimizing the portfolio, the subportfolios can trade with one another, thus avoiding transaction costs for the fund. Currently, GMO uses this approach to construct and manage eleven different portfolios representing over $8 billion in assets. The benefits to GMO of the implementation of this mixed-integer-programming process include:

1. retaining the existing client base,
2. opening new growth opportunities,
3. reducing the number of distinct stocks held in the portfolios by an average of 48.7%,
4. reducing the number of stock transactions by an average of 79.3%, and
5. as a result of points 3 and 4, reducing the annual cost of trading the portfolios by $4 million.

Bertsimas, D., Darnell, C. and Soucy, R. (1999). Portfolio construction through mixed-integer programming at Grantham, Mayo, Van Otterloo and Company. *INTERFACES, 29*(1), 49-66.

Seattle Mariners Season Tickets. Seven friends formed a consortium to purchase a pair of Seattle Mariners season tickets for 1996. To assign the game tickets, each participant rank-ordered each of the 81 games. A computer program was then developed that assigned to each participant, in turn, the game he had ranked highest of those not yet assigned. The problem with this scheme was that participants tended to rank games in clusters that reflected their highest priorities, because the games were rank-ordered only once. For example, one of the participants was originally from New York City, so he gave all of the games when the N. Y. Yankees visited Seattle the highest ranking. As might be expected, he received the tickets for all four of those games. As a result of this and similar rankings, the assignment of tickets to individual participants was often not well distributed across the entire season. Post-assignment trades of tickets among some of the participants subsequently alleviated this problem.

For the 1997 season, the consortium added one participant and decided to purchase four season tickets. These two factors greatly complicated the problem of allocating the tickets.

To solve this more complicated problem, a large number of 0-1 integer decision variables were defined based on whether or not a particular individual was assigned one or both pairs of tickets to a particular game. Rules were then formulated in an attempt to ensure that each participant received the mix of tickets he desired. Finally, each participant rank-ordered each game, so that a score to measure his level of satisfaction with a particular assignment of tickets could be computed. Putting this all together, the available tickets were allocated so as to minimize the dissatisfaction of the participant who was most dissatisfied.

Overall, the participants were very satisfied with the way this scheme worked in 1997, and the consortium planned to use the same system, with one minor adjustment, in 1998. All of the participants planned to again be part of the consortium in 1998, and there was even a waiting list for others who had asked to join the consortium.

Grandine, T. (1998). Assigning season tickets fairly. *INTERFACES, 28*(4), 15-20.

Army Base Realignment and Closure: The United States Army has downsized its forces by more than 30% since 1989. These reductions were accompanied by a detailed plan to eliminate and/or realign dozens of military bases around the world. The closure or realignment of a military base affects the military's ability to deploy personnel and equipment rapidly and also has a major impact on the local economy surrounding the base. Due to the sensitive nature of these decisions and their political impact, Congress passed special legislation governing the decision-making process. A 1995 commission decided to close 29 bases and realign 11 others. Approximately $2 billion was budgeted

to implement the decisions with anticipated long-term annual operating savings of $360 million.

A mixed-integer linear programming model was developed to assist the military in scheduling the sequence of closures and realignment over a six-year period while staying within the $2 billion budget. The model's name was BRACAS (Base Realignment and Closure Action Scheduler). The model develops timetables for actions that meet annual budgetary constraints and maximizes the net present value of the plan. 0-1 decision variables were used to represent whether or not in a specific year:

- ❑ a significant number of personnel were removed from a base,
- ❑ construction of a new facility was begun, and
- ❑ all actions related to a specific base were completed.

During the first application of BRACAS, the schedule it suggested yielded $400 million in savings when compared to the original plan. Major General Robert T. Howard, Deputy Assistant Secretary of the Army for Budget, questioned the results and asked for changes in the assumptions of the model. These model changes were made within an hour and the results were even better than before. This experience was critical in obtaining top-level credibility for BRACAS as a planning tool that could answer a wide range of "what-if" questions. One direct impact of the model was to increase the FY97 allocation by $100 million in order to produce a $233 million dollar savings over the six-year period.

Dell, R. F. (1998), Optimizing army base realignment and closure, *INTERFACES 28*(6), 1-18.

Tata Steel of India: Tata Iron and Steel Company is one of the largest companies in India. In the early 1990's it reported revenues in excess of $1.2 billion and profits of $160 million. Tata's 75,000 workers annually produced 2.4 million tons. In 1984, Tata Iron and Steel completed a major modernization effort that enhanced steel making capacity. Work began on a mixed-integer linear programming (MILP) model that would optimally allocate scarce resources so as to maximize profits. This emphasis on optimal profits represented a significant change in thinking. Up to that time the spotlight had always been on maximizing tonnage without paying too much attention to that policy's overall impact on profit. It was wrongly assumed that the two–tonnage and profit–were almost equivalent.

Tata Steel is a fully integrated iron and steel making plant that consumes vast amounts of energy. An early focus of the modeling effort was the impact of scarce power resources on production planning. In metropolitan areas, power availability for Tata Steel fluctuated by time of day and season and in some instances the reduction of power was abrupt. One critical set of decisions involved determining which production activities to shut down as power resources declined. Before the model had been implemented, the policy was to shut down finishing mills first and later primary mills as power became scarcer. The model demonstrated that the optimal strategy was really to shut only a few

of the finishing mills before starting to turn off the primary mills. The decisions as to which mills to turn off were represented in the model as 0-1 variables.

The first major success of the model occurred in November 1986. The model enabled Tata Steel to generate more profits than the month before even though there was less available power and total production was down. It is estimated that in the first year of implementation, the MILP model increased Tata Steel's profits by $73 million as it shifted the company's focus to maximizing profit rather than tonnage. The model has since been applied to other parts of the company and its affiliates that produce tubes, tinplate and steel wire rope.

Sinha, G. P., Chandrasekaran, B. S., Mitter, N., Dutta, G., Singh, S. B., Choudhury, A. R., and Roy, P. N. (1995), Strategic and operational management with optimization at Tata Steel, *INTERFACES 25*(1), 6-19.

Objectives of Module

Students who use this module should have some familiarity with spreadsheet software. If many do not, the module is still useable if the teacher carefully sequences items 3-7 in Part 2 of the Student Activity, so as to develop students' familiarity with the basic spreadsheet functions that are required. The heart of the mathematics in the module lies in the problem formulation. Then the spreadsheet technology is used to solve the problem efficiently. Specific objectives of the module include the following:

1. Represent a set of yes-no decisions using 0-1 integer decision variables.
2. Use a spreadsheet to structure a mathematical problem.
3. Manipulate a spreadsheet to efficiently solve a mathematical problem.
4. Integrate 0-1 variables into the formulation of constraints and objectives.
5. An extension combines continuous variables with 0-1 variables in a single formulation.

Initiating Activity

The student activity investigates different investment plans for a modest sum of money. Since many high school students may be unfamiliar with many aspects of investments, you may prefer to open an initiating dialogue with the class that discusses the process of investing money. Some issues you might address in this class discussion are:
1. Why do people invest their money?
2. How do you compare investments? (e.g., percent return or total dollar return)
3. Make a list of personal reasons for investing at this time in your life.
4. Provide a situation where investing would be a bad idea.

Teaching Notes

Students may question why each investment requires a specific fixed investment to receive an annual return. Some reasons why this might be so include:

❑ One may contribute to a project with a one-time start-up cost.
❑ A bond may come in a fixed denomination. (This must not be confused with an annual return that is compounded over time.)
❑ A company may need startup funds so they may offer a one-time deal to investors with a specific guaranteed annual return.

These opportunities would only be available once and would require investing a specific amount of money.

Part 1: Developing an Investment Plan. In item 4, students are asked to discuss informally the difference between feasible and infeasible solutions. A feasible solution to a problem is a set of values of the decision variables (the amount of money placed in each investment in this case) that meets all of the constraints of the problem situation (the sum of the monies placed in each investment must be less than or equal to $12,000 in this case).

For item 7, even if students all choose to investigate the optimal plan, you will still need to consider all of the possible options in order to ensure that the plan chosen really is optimal.

Part 2: Latisha Develops an Investment Plan. For item 1, you may need to interpret "APR" as "Annual Percentage Rate" for students.

When students have completed item 6 and before they attempt item 7, you may want to ensure that they are familiar with the sum function of the spreadsheet. Alternatively, they could define cells E6 and F6 using an equation (e.g., E6=E2+E3+E4+E5).

As students complete a table similar to the one between items 8 and 9, you may want to point out to students that the table in the student activity is a sample. There are more than two additional plans that will need to be considered.

Items 18-25 require students to formulate the problem by defining a 0-1 integer variable for each of the individual options. We use the language of (integer) linear programming here. You may need to discuss with students the meaning of "objective function" and "constraints". The objective function defines a quantity that one would like to optimize (maximize or minimize). In this case, since we are interested in maximizing the total annual return, it will be the objective function. The objective function is then defined in terms of decision variables. In this case, the decision variables represent whether or not to invest in each option. Constraints are limitations placed on functions of the decision variables. In this case, the total amount invested is constrained by the total amount available.

Extension. One extension is provided in which the excess funds not invested will be placed in a CD account. Students could adjust a spreadsheet to accommodate the CD. Note that the APR .023 could be stored in a remote cell such as A10. This allows one to change the APR and assess its impact on the optimal solution. Because all of the funds will be invested in this scenario, the sum of all of the investments must now equal $15,000, and the constraint is now an equality.

Homework. Two homework problems are provided. The first involves the assignment of four swimmers to the four legs of a medley relay. The second is called a set coverage problem. In this case, students are asked to determine which three of four schools to open for evening activities so as to ensure that the maximum distance anyone would need to travel does not exceed a given distance.

Project Idea

Identify a community decision, real or imagined, that involves the placement of a network of facilities. The decision could involve neighborhood parks, recreation centers, fire stations, health clinics, schools or government offices.
- ❑ Identify a set of possible locations out of which a subset will be chosen.
- ❑ What measure(s) would you use to compare alternative networks of facilities?
- ❑ What data would you collect to help make the decision?

Answer Key

Student Activity Part 1:

1. No, there is no combination of investments that totals = $12,000.
2. Each investment requires a fixed amount. There is a guaranteed annual return for each investment.
3. Answers vary; for example, invest in Dynamite Dollars ($6,900) and Super Stash ($4,300), for a total investment of $11,200.
4. A plan is feasible if the sum of the required investments is less than or equal to $12,000, infeasible if the sum of the required investments is greater than $12,000.
5. Answers vary. For example, investing in Dynamite Dollars and Super Stash has an annual return of $518 + $285 = $803.
6. Answers vary.
7. Make none of the investments (students might not consider this possibility), make only one of the individual investments (three possibilities), and make any two of the investments (also three possibilities). There are a total of seven feasible alternatives. Note that, including the one infeasible option, there are 8 potential plans. You might want to ask students why that makes sense.
8. One way is to list all of the possible patterns and identify every plan for each pattern. For example, one pattern involves all pairs of investments. Another pattern is choosing only one investment.
9. Cold Cash ($294) and Dynamite Dollars ($518).
10. There will be a significant increase in the number of plans. If students have noticed in item 7 that the number of potential plans for 3 investments is 2^3, you might ask them how many potential investment plans there would be.

Student Activity Part 2:

1. $345
2. Answers vary; at this point, do not worry whether the option students choose is feasible.
3. It is feasible if the sum of the investments made is less than or equal to $15,000.
4. This corresponds to investing in all four funds.
5. Multiply the cells in column B by the corresponding cells in column D. Then calculate the sum of column E.
6. The total is $22,800, which is more than the $15,000 available for investment.
7. No, it is not feasible since it exceeds the available dollars for investment.
8. Invest in Cold Cash, Dynamite Dollars, and Super Stash.
9. Carefully, reposition the lone zero in the cell corresponding to the plan not invested in. In every case the column should contain one 0 and three 1s.
10. The optimal solution invests in Fabulous Funds and Dynamite Dollars. The total return is $1,106. A close second is the investment plan that includes Cold Cash, Dynamite Dollars, and Super Stash and yields $1,097.
11. Answers vary.
12. c

13. The students should have found that every investment plan involving exactly two funds is feasible. Therefore, any plan involving exactly one fund must also be feasible and have a lower yield. The only other alternative is not investing in any fund.
14. The goal is to find the optimal feasible plan.
15. There are $2^5 = 32$ possibilities. You can either invest or not invest in each fund. Some students may miss the possibility of not investing at all.
16. The number of combinations of investments grows exponentially.
17. A spreadsheet enables you to evaluate the feasibility and yield of each new plan by making just a few changes.
18. 0 and 7800; 7800*f* represents the number of dollars invested in Fabulous Funds.
19. 0 and 588; 588*f* represents the number of dollars of annual return from Fabulous Funds.
20. $A = 588f + 294c + 518d + 285s$
21. $7900f + 3700c + 6900d + 4300s$
22. There is a 15000 limit on this total.
23. $7900f + 3700c + 6900d + 4300s = 15000$
24. With only a few decision variables, it was easy to manipulate the spreadsheet.
25. With a large number of decision variables, even with a spreadsheet, it would be too difficult to keep track of the required manipulations by hand. However, this formulation would allow one to use other software to test all of the possibilities.

Extension:

1. x is not a 0-1 decision variable.
2. $A = 588f + 294c + 518d + 285s + .023x$
3. $7900f + 3700c + 6900d + 4300s + x = 15000$
4. It is an equality.
5. Optimal Solution:

investment	amount invested	annual income	0-1 variable	actual investment	actual income
Fabulous Funds	7900	588	1	7900	588
Cold Cash	3700	294	0	0	0
Dynamite Dollars	6900	518	1	6900	518
Super Stash	4300	285	0	0	0
2.3% CD	200	4.60		200	4.60
Totals				15000	1110.60

Homework

1. Assigning Swimmers to the Medley Relay:

1. 4!=24
2. A 1 means that a swimmer was assigned to a particular leg of the relay.
3. The 1s are the row and column sums and ensure that each swimmer is assigned to one and only one leg of the relay, and each leg is assigned to one and only one swimmer.
4. Their total average time, 208.7.
5. Each swimmer is assigned to exactly one leg of the relay, and each leg of the relay is assigned to exactly one swimmer.
6. These are the average times for that event for the particular swimmer.
7. This is the total average time for one particular assignment of swimmers to the legs of the relay.
8. Each cell of Table 3 is the product of the corresponding cells in Tables 1 and 2.
9. Smith: back, Braun: free, Cisek: fly, Rivera: breast; Total average time = 203.5 sec.
10. There would be five times as many possibilities. Then the number of possibilities would be $5(4)(3)(2) = 120$. Note that one swimmer would not be assigned to any leg of the relay.

2. Opening School Facilities:

1. A 1 means that the entire region is within 2.5 miles of the school. A 0 means that the entire region is not within 2.5 miles of the school.
2. Adams and Washington.
3. No.
4. Yes. Regions 4 and 12.
5. Both Jefferson and Madison must be opened.
6. Yes; Jefferson, Madison, and Adams or Jefferson, Madison, and Washington.
7. 3.5 miles.
8. Any two of the four schools can meet a 3.5 mile criterion.

Blackline Masters

	A	B	C	D	E	F
1	Investment	Amount Invested	Annual Income	0-1 decision variable	Actual Investment	Actual Income
2	Fabulous Funds	7900	588			
3	Cold Cash	3700	294			
4	Dynamite Dollars	6900	518			
5	Super Stash	4300	285			
6				Is this sum feasible? ➔		

Latisha Develops an Investment Plan

	A	B	C	D	E	F
1	**investment**	**amount invested**	**annual income**	**0-1 variable**	**actual investment**	**actual income**
2	Fabulous Funds	7900	588	1	7900	588
3	Cold Cash	3700	294	0	0	0
4	Dynamite Dollars	6900	518	1	6900	518
5	Super Stash	4300	285	0	0	0
6	2.3% Savings	200	4.60		200	4.60
7					15000	1110.60
8						
9	Savings rate					
10	2.3%					

CD Extension

Table 1

	free	back	breast	fly
Smith	47.59	51.67	58.31	49.18
Braun	47.31	51.62	58.95	49.89
Cisek	48.29	52.59	58.92	49.13
Rivera	48.53	53.34	59.38	50.59

Table 2

	free	back	breast	fly	
Smith	1	0	0	0	1
Braun	0	1	0	0	1
Cisek	0	0	1	0	1
Rivera	0	0	0	1	1
	1	1	1	1	

Table 3

	free	back	breast	fly
Smith	47.59	0	0	0
Braun	0	51.62	0	0
Cisek	0	0	58.92	0
Rivera	0	0	0	50.59
				208.7

Assigning Swimmers to the Medley Relay

Table 4: Distances from Regions to Schools

REGION

	1	2	3	4	5	6	7	8	9	10	11	12
Adams H.S.	1.2	1.3	2.3	3.3	1.5	1.6	2.6	3.6	2.5	2.6	3.6	4.6
Jefferson H.S.	3.1	2.1	1.1	2.2	3.3	2.3	1.2	2.2	4.3	3.3	2.2	3.2
Madison H.S.	4.5	3.5	2.3	3.4	3.5	2.5	1.3	2.4	3.3	2.3	1.2	2.4
Washington H.S.	2.4	2.5	3.5	4.5	1.4	1.5	2.5	3.5	1.5	1.4	2.4	3.4

Table 5: 2.5 Mile Criterion

REGION

	1	2	3	4	5	6	7	8	9	10	11	12
Adams H.S.	1	1	1	0	1	1	0	0	1	0	0	0
Jefferson H.S.	0	1	1	1	0	1	1	1	0	0	1	0
Madison H.S.	0	0	1	0	0	1	1	1	0	1	1	1
Washington H.S.	1	1	0	0	1	1	1	0	1	1	1	0

Open School Facilities